

TRANSLATION-LIKE ACTIONS YIELD REGULAR MAPS

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ABSTRACT. For finitely generated groups H and G , we observe that H admits a translation-like action on G implies there is a regular map, which was introduced in Benjamini, Schramm and Timár's joint paper, from H to G . Combining with several known obstructions to the existence of regular maps, we have various applications. For example, we show that the Baumslag-Solitar groups do not admit translation-like actions on the classical lamplighter group.

1. INTRODUCTION

The concept of translation-like actions was introduced by Whyte to solve a geometric version of the von Neumann conjecture [19]. It serves as a geometric analogy of subgroup containment for finitely generated groups: if H is a subgroup of a finitely generated group G , then the natural right action of H on G is a translation-like action. Later on, answering a question of Whyte, Seward solved a geometric version of Burnside's problem which is formulated using translation-like actions [17]. Recently, translation-like actions play an important role in studying the question which finitely generated groups admit a weakly aperiodic shift of finite type [13]. Similar approaches (using notions from geometric group theory) to this question also appeared in [4, 6].

Despite the above success of application of this concept in various problems, it seems to us that not too much is known on whether a group H admits a translation-like action on another finitely generated group G assuming H is not a subgroup of G . For known examples, see [5, 13, 17, 19]. The purpose of this note is to observe that translation-like actions yield regular maps, which was introduced in [3].

Theorem 1.1 (Main theorem). *Let G and H be finitely generated groups. If H admits a translation-like action on G , then $H \rightarrow_{reg} G$.*

It is well-known that there are several ways to rule out existence of regular maps between spaces: asymptotic dimension, Dirichlet harmonic functions, growth and separation (see [3, last paragraph in section 1]). By the above theorem, we also get obstructions to admitting translation-like actions for a pair of groups. We discuss these in Section 3.

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2. DEFINITIONS AND PROOF OF THE MAIN THEOREM

First, we recall the general definition of translation-like actions following Seward [17, Definition 1.1], but here we use left actions. Note that the original definition is due to Whyte [19, Definition 6.1].

Definition 2.1 (Translation-like actions). *Let H be a group and let (X, d) be a metric space. A left action $*$ of H on X is translation-like if it satisfies the following two conditions:*

- (i) *The action is free (i.e. $h * x = x$ implies $h = 1_H$).*
- (ii) *For every $h \in H$ the set $\{d(x, h * x) \mid x \in X\}$ is bounded.*

Next, we recall the definition of regular maps following [3].

Definition 2.2 (Regular maps). *Let $(Z, d_Z), (X, d_X)$ be metric spaces, and let $\kappa < \infty$. A (not necessarily continuous) map $f : Z \rightarrow X$ is κ -regular if the following two conditions are satisfied.*

- (i) *$d_X(f(z_0), f(z_1)) \leq \kappa(1 + d_Z(z_0, z_1))$ holds for every $z_0, z_1 \in Z$;*
- (ii) *For every open ball $B = B(x_0, 1)$ with radius 1 in X , the inverse image $f^{-1}(B)$ can be covered by κ open balls of radius 1 in Z .*

A regular map is a map which is κ -regular for some finite κ . Write $Z \rightarrow_{reg} X$ if there is a regular map from Z to X .

Note that as mentioned in [3, p. 5], if there is a quasi-isometry between bounded degree graphs Z and X , then there is a regular map from Z to X and also the other direction. Hence being quasi-isometric implies $Z \rightarrow_{reg} X$ and $X \rightarrow_{reg} Z$, but the converse does not hold.

From now on, we fix a finitely generated group H and take (X, d) to be a finitely generated group G using a left-invariant word length metric d associated to some finite symmetric generating set T , i.e. $d_G(x, y) := \ell_G(x^{-1}y)$ for all $x, y \in G$, where ℓ_G is the word length associated to T .

If $H \curvearrowright^* G$ is a translation-like action, then we can define a map $L : H \times G \rightarrow G$ by setting $L(h, x) = x^{-1}(h * x)$ for all $x \in G, h \in H$.

Note that \curvearrowright^* is an action implies that $L(h_1, x)L(h_2, h_1 * x) = L(h_2 h_1, x)$ for all $h_1, h_2 \in H$ and $x \in G$.

Write $c(g, x) := L(g, x)^{-1}$, then $c : H \times G \rightarrow G$ is a cocycle in the usual sense, i.e. $c(h_1 h_2, x) = c(h_1, h_2 * x)c(h_2, x)$ for all $h_1, h_2 \in H, x \in G$. And the two conditions in the definition of translation-like action are just the following.

- (1) For all $x \in G$, the map $H \ni h \mapsto c(h, x) \in G$ is 1-1.
- (2) For all $h \in H$, the set $\{c(h, x) : x \in G\}$ is bounded, i.e. $\sup_{x \in G} \ell_G(c(h, x)) =: \lambda_h < \infty$.

We are ready to prove our main theorem.

Proof of Theorem 1.1. Fix any $x \in G$, define $f : H \rightarrow G$ by $f(h) := c(h^{-1}, x)^{-1}$. We claim this map is a regular map.

Fix a symmetric generating set S for H and define $\kappa := \max_{s \in S} \lambda_s + \#T$, where $\lambda_h := \sup_{x \in G} \ell_G(c(h, x))$ for all $h \in H$.

Note that the second condition in Definition 2.2 is clear since $h \mapsto c(h, x)$ is 1-1. We are left to check the first condition holds.

$$\begin{aligned}
& d_G(f(h_1), f(h_2)) \\
&= d_G(c(h_1^{-1}, x)^{-1}, c(h_2^{-1}, x)^{-1}) \\
&= \ell_G(c(h_1^{-1}, x)c(h_2^{-1}, x)^{-1}) \\
&= \ell_G(c(h_1^{-1}, x)c(h_2^{-1} * x)) \\
&= \ell_G(c(h_1^{-1}h_2, h_2^{-1} * x)) \\
&= \ell_G(c(s_1 \cdots s_k, h_2^{-1} * x)), \text{ write } h_1^{-1}h_2 = s_1 \cdots s_k, \text{ where } k = \ell_H(h_1^{-1}h_2), s_i \in S. \\
&= \ell_G(c(s_1, s_2 \cdots s_k h_2^{-1} * x) \cdots c(s_k, h_2^{-1} * x)) \\
&\leq \sum_{i=1}^k \ell_G(c(s_i, s_{i+1} \cdots s_k h_2^{-1} * x)) \\
&\leq \kappa \ell_H(h_1^{-1}h_2) = \kappa d_H(h_1, h_2).
\end{aligned}$$

□

Remark 2.3. *Similar calculation was already used in [15, 16]. In fact, there is a connection between translation-like actions and continuous orbit equivalence theory, which may be worth exploring further.*

3. APPLICATIONS

As mentioned in the introduction, we use the obstruction to existence of regular maps, i.e. asymptotic dimension, which was first defined by Gromov [10] (see [1] for a nice survey), Dirichlet harmonic functions [2], growth and separation [3], to deduce results on non-existence of translation-like actions. Conversely, we may deduce results on the existence of regular maps between two groups. The above process would generate many (counter)examples, we just list a few of them here and we refer the readers to the above papers for definitions of the above invariants.

Corollary 3.1. *If G is a finitely generated non-amenable group, then $F_2 \rightarrow_{\text{reg}} G$, where F_2 is the non-abelian free group on two generators.*

Proof. This is clear by Whyte's solution to the geometric von Neumann conjecture, see [19]. □

Corollary 3.2. *If G is a finitely generated infinite group, then $\mathbb{Z} \rightarrow_{\text{reg}} G$.*

Proof. This is clear by Seward's solution to the geometric Burnside's problem, see [17]. □

Corollary 3.3. *\mathbb{Z}^d does not admit translate-like actions on F_2 for all $d \geq 2$.*

Proof. By theorem 1.1, we just need to show $\mathbb{Z}^d \not\curvearrowright_{reg} F_2$ if $d \geq 2$. By [3, Corollary 3.3], $\lim_{n \rightarrow \infty} sep_{\mathbb{Z}^d}(n) = \infty$ if $d \geq 2$, while $sep_{F_2}(\cdot)$ is bounded by [3, Section 2]. Since the separation function is monotone non-decreasing with respect to regular maps by [3, Lemma 1.3], we deduce $\mathbb{Z}^d \not\curvearrowright_{reg} F_2$. \square

Corollary 3.3 answers a question of Cohen, see [5, p.4]. This question is one initial motivation for us to look at translation-like actions.

Corollary 3.4. *The Baumslag-Solitar group $BS(m, n) := \langle s, t \mid st^m s^{-1} = t^n \rangle$ does not admit translate-like actions on the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ for any integers m, n .*

Proof. Case 1: Assume $mn \neq 0$.

It suffices to show $BS(m, n) \not\curvearrowright_{reg} (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ if $mn \neq 0$.

First, $asdim((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}) = 1$ by [9, Proposition on p.5]. Then note that the asymptotic dimension is monotone non-decreasing under regular maps, see [3, Section 6] or just apply [1, Theorem 29] in our setting. Therefore, we are left to show $asdim(BS(m, n)) \geq 2$ if $mn \neq 0$.

First, $BS(m, n)$ is always an infinite group, hence $asdim(BS(m, n)) > 0$ by [9, Lemma 1] or Corollary 3.2. We are left to show $asdim(BS(m, n)) \neq 1$. By [8, Corollary 1.2] (see also [7, 9, 12]), we just need to check $BS(m, n)$ does not contain free group as a subgroup of finite index. This is clear since $BS(m, n)$ is torsion-free when $mn \neq 0$ by [14] and any torsion-free virtually free groups are free groups by Stallings' work [18].

Case 2: Assume $mn = 0$.

$BS(m, n) \cong \mathbb{Z} * (\mathbb{Z}/n\mathbb{Z})$ or $\mathbb{Z} * (\mathbb{Z}/m\mathbb{Z})$. This group contains a free group, then it does not admit a translation-like action on $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ by [13, Lemma 1.3] and [19, Theorem 6.2]. \square

Remark 3.5. *Note that to prove the above corollary, we can directly focus on $BS(1, n)$, which is amenable. Also note that if $m = n = 1$, $BS(1, 1) = \mathbb{Z}^2$, then we can also deduce $\mathbb{Z}^2 \not\curvearrowright_{reg} (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ by [3, Proposition 6.1]. If $m = 0$ or $n = 0$, then $BS(m, n) \cong \mathbb{Z} * (\mathbb{Z}/n\mathbb{Z})$ or $\mathbb{Z} * (\mathbb{Z}/m\mathbb{Z})$, this group has asymptotic dimension one by [1, Theorem 84].*

Corollary 3.4 answers [13, Conjecture 3] negatively. It also answers the geometric Gersten problem stated in [17] in the negative. The original geometric Gersten problem (stated under a further assumption compared with the one stated in [17]) was asked by Whyte in [19, p.107]. See [5] for more discussion on this conjecture. Note that the lamplighter group is a potential counterexample to the above conjectures was already suggested in [13].

We end the paper with the following question:

Question 3.6. *Does [11, Theorem 2.3] still hold for regular maps (maybe up to some modification of the definition of fat bigons there)?*

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